## Notes on Basefold (Part I): Foldable Linear Codes

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Basefold can be regarded as an extension of FRI, thereby supporting Proximity Proofs and Evaluation Arguments for Multi-linear Polynomials. Compared to Libra-PCS, Hyrax-PCS, and Virgo-PCS, Basefold does not rely on the MLE Quotients Equation to prove the value of an MLE at the point **u**:

$$ilde{f}(X_0, X_1, X_2, \dots, X_{n-1}) = \sum_{i_0=0}^{n-1} (X_i - u_i) \cdot q_i(X_0, X_1, X_2, \dots, X_{i-1})$$
 (1)

Instead, Basefold leverages the Sumcheck protocol to reduce the value of  $\tilde{f}(\mathbf{X})$  at a pre-specified point to its value at a random point  $\vec{\alpha}$ . The latter can then utilize an FRI-like approach, where the verifier provides a challenge vector  $\vec{\alpha}$  to recursively fold  $\tilde{f}(\mathbf{X})$ . This approach allows the prover to simultaneously demonstrate an upper bound on the degree of  $\tilde{f}(\mathbf{X})$  (Proof of Proximity) and, based on the characteristics of MLE, prove the value of  $\tilde{f}(\vec{\alpha})$ . By combining the Sumcheck protocol with an FRI-style folding protocol, we elegantly obtain an Evaluation Argument for an MLE.

Another significant insight of Basefold is how to apply an FRI-style Proof of Proximity over arbitrary finite fields. It is known that the FRI protocol fully utilizes the structure of Algebraic FFTs over finite fields and performs interactive folding, which allows the codeword length to be exponentially reduced while maintaining the minimum relative Hamming distance bound, thereby enabling the verifier to easily verify a folded short codeword. However, for general finite fields where constructing FFTs is challenging, the FRI protocol cannot be directly applied.

Basefold introduces the concept of Random Foldable Codes. This is a framework for encoding using a recursive approach, where recursive encoding can be seen as the inverse process of recursive folding. In the Commit phase of the Basefold protocol, the prover first encodes each segment of the message using a base encoding scheme  $G_0$ , then encodes these codewords pairwise (a process similar to the butterfly operation in FFT computations), and finally obtains a single codeword. In the Commit-phase stage, the prover and verifier interactively perform half-folding on the committed single codeword and then commit to the half-folded codeword. It can be proven that the Relative Hamming Distance of this half-folded codeword remains above a clear lower bound. The parties then continue folding until the length is reduced to that of  $G_0$ -encoded length. In this way, both parties obtain a series of committed codewords. Note that since the encoding process is performed recursively, the codeword possesses folding capabilities similar to those of RS Code encoding. The advantage of this approach is evident: using Recursive Folding FRI protocols results in significantly smaller proof sizes compared to protocols like Tensor Codes. Additionally, Basefold's technique can be applied to any finite field ( $|\mathbb{F}|>2^{10}$ ), including extension fields  $\mathbb{F}_{p^k}$ . Naturally, similar to the FRI protocol, the commitment computation in the Basefold protocol requires  $O(N \log(N))$  time complexity, while the prover's computational effort in the subsequent proof process remains at O(N). If a Merkle Tree is used as the commitment scheme for the codeword, the proof size complexity is  $O(\log^2(N))$ .

In this article, we first explore the concept of Foldable Linear Codes.

## What are Foldable Linear Codes

Suppose there is a linear code  $C_0: \mathbb{F}_p^{k_0} \to \mathbb{F}_p^{n_0}$  based on the finite field  $\mathbb{F}_p$ , where the message length is  $k_0$ , and the codeword length is  $n_0$ . Assume that the encoded length is amplified by a factor of R compared to the message length, i.e., 1/R is the traditional code rate. According to the definition of linear codes, there must exist an encoding matrix  $G_0: \mathbb{F}_p^{k_0 \times n_0}$  such that

$$\mathbf{m}G_0 = \mathbf{c} \tag{2}$$

Then, we can construct a new encoding matrix  $G_1$  based on the encoding matrix  $G_0$ :

$$G_1 = \begin{bmatrix} G_0 & G_0 \\ G_0 \cdot T_0 & G_0 \cdot T_0' \end{bmatrix}$$
(3)

where  $T_0$  and  $T'_0$  are two diagonal matrices with diagonal elements set to the parameters of the encoding scheme, which we will explain later.

If we consider  $G_1$  as the encoding matrix of another linear code  $C_1$ , then the parameters of  $C_1$  are  $[R, 2k_0, 2n_0]$ . That is, compared to  $C_0$ , the code rate of  $C_1$  remains unchanged, but the message and codeword lengths are both doubled.

For example, suppose  $\mathbf{m} = (m_0, m_1, m_2, m_3)$ , and the base encoding scheme  $C_0$  is a simple Repetition Code with  $k_0 = 2$  and  $n_0 = 4$ . Thus, the message length of  $\mathbf{m}$  exactly meets the requirements of  $G_1$ , i.e.,  $k_1 = 2 \cdot k_0 = 4$  and the encoded length is  $n_1 = 2 \cdot n_0 = 8$ .

The encoding matrix  $G_0$  of the base code  $C_0$  is defined as:

$$G_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
(4)

Assume the diagonal elements of the system parameter  $T_0$  are  $(t_0, t_1, t_2, t_3)$ , and those of  $T'_0$  are  $(t'_0, t'_1, t'_2, t'_3)$ . We construct  $G_1$  using the formula above:

$$G_{1} = \begin{bmatrix} 1 & 0 & 1 & 0 & | & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ t_{0} & 0 & t_{2} & 0 & t'_{0} & 0 & t'_{2} & 0 \\ 0 & t_{1} & 0 & t_{3} & | & 0 & t'_{1} & 0 & t'_{3} \end{bmatrix}$$
(5)

Substituting  ${f m}$  directly, we obtain:

$$\mathbf{m}G_{1} = \begin{bmatrix} m_{0} & m_{1} & m_{2} & m_{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ t_{0} & 0 & t_{2} & 0 & t'_{0} & 0 & t'_{2} & 0 \\ 0 & t_{1} & 0 & t_{3} & 0 & t'_{1} & 0 & t'_{3} \end{bmatrix}$$
(6)

We list the encoded codeword vector separately:

Directly performing matrix operations, the structure of this encoding is not very clear. Let us approach it differently: the message  $\mathbf{m}$  can be split into two equal-length parts, the left  $\mathbf{m}_l = (m_0, m_1)$  and the right  $\mathbf{m}_r = (m_2, m_3)$ ,

$$\mathbf{m}G_1 = (\mathbf{m}_l \parallel \mathbf{m}_r)G_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{m}_l & 0 \\ 0 & \mathbf{m}_r \end{bmatrix} \begin{bmatrix} G_0 & G_0 \\ G_0 \cdot T_0 & G_0 \cdot T_0' \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{m}_l G_0 & \mathbf{m}_l G_0 \\ \mathbf{m}_r G_0 T_0 & \mathbf{m}_r G_0 T_0' \end{bmatrix}$$
(8)

For the  $2 \times 2$  matrix on the right side of the equation, we first compute the top-left and top-right submatrices, which are identical submatrices obtained by encoding  $\mathbf{m}_l$  with  $G_0$ :

$$\mathbf{m}_{l}G_{0} = \begin{bmatrix} m_{0} & m_{1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{0} & m_{1} & m_{0} & m_{1} \end{bmatrix}$$
(9)

And the bottom-left submatrix is:

$$\mathbf{m}_{r}(G_{0}T_{0}) = (\mathbf{m}_{r}G_{0})T_{0} = \begin{bmatrix} m_{2} & m_{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} T_{0}$$

$$= \begin{bmatrix} m_{2} & m_{3} & m_{2} & m_{3} \end{bmatrix} \begin{bmatrix} t_{0} & 0 & 0 & 0 \\ 0 & t_{1} & 0 & 0 \\ 0 & 0 & t_{2} & 0 \\ 0 & 0 & 0 & t_{3} \end{bmatrix}$$

$$= (t_{0}m_{2}, t_{1}m_{3}, t_{2}m_{2}, t_{3}m_{3})$$
(10)

The result for  $\mathbf{m}_r(G_0T'_0)$  is almost the same, with  $t_i$  replaced by  $t'_i$ :

$$\mathbf{m}_r(G_0 T_0') = (t_0' m_2, t_1' m_3, t_2' m_2, t_3' m_3)$$
(11)

It is easy to verify that we obtain the same result for  $\mathbf{m}$  encoded with  $G_1$ .

We can simplify this computation process with the following equation:

$$\mathbf{m}G_1 = \mathbf{m}_l G_0 + (t_0, t_1, t_2, t_3) \circ \mathbf{m}_r G_0 \parallel \mathbf{m}_l G_0 + (t_0', t_1', t_2', t_3') \circ \mathbf{m}_r G_0$$
(12)

If we consider  $k_0$  as a system parameter (constant), the encoding computation in the above equation has a complexity of  $O(n_0)$ , including two  $G_0$  encodings and two component-wise additions of length  $n_0$ . Additionally, this equation resembles the butterfly operation in (Multiplicative) FFT algorithms:

$$a' = a + t \cdot b, \quad b' = a - t \cdot b$$

$$\tag{13}$$

In fact, as illustrated later, the encoding process of Foldable Codes is a generalized extension of RS-Code.

We can further recursively construct  $G_2, G_3, \ldots, G_d$ , eventually obtaining a linear code  $C_d : \mathbb{F}_p^{k_0} \to \mathbb{F}_p^n$ , where the message length is  $k = k_0 \cdot 2^d$ , and the encoding length is  $n = c \cdot k_0 \cdot 2^d$ . The code rate is  $\rho = \frac{1}{R}$ , where  $k_0$  is the message length of the base encoding,  $n_0$  is the base encoding length, and the choice of the base encoding is highly flexible.

We use the symbol  $G_d$  to denote the encoding matrix (or generator matrix) of  $C_d$ ,  $G_d \in \mathbb{F}_p^{k \times n}$ . For  $i \in \{1, 2, ..., d\}$ , we have the following recursive relation:

$$G_{i} = \begin{bmatrix} G_{i-1} & G_{i-1} \\ G_{i-1} \cdot T_{i-1} & G_{i-1} \cdot T_{i-1}' \end{bmatrix}$$
(14)

Here,  $T_{i-1}$  and  $T'_{i-1}$  are two diagonal matrices,

$$T_{i-1} = \begin{bmatrix} t_{i-1,0} & 0 & \cdots & 0 \\ 0 & t_{i-1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{i-1,n_{i-1}} \end{bmatrix}, \quad T'_{i-1} = \begin{bmatrix} t'_{i-1,0} & 0 & \cdots & 0 \\ 0 & t'_{i-1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t'_{i-1,n_{i-1}} \end{bmatrix}, \quad (15)$$

and their diagonal elements at the same positions are distinct:

$$orall j \in [0, n_i - 1], t_{i,j} 
eq t'_{i,j}$$
 (16)

Assuming we can compute  $G_d$  using the above recursive equation, then for a message  $\mathbf{m}_d$  of length k, the encoding process can be directly calculated as follows:

$$\mathbf{w}_d = \mathbf{m}_d G_d \tag{17}$$

However, encoding directly using the generator matrix  $G_d$  is relatively inefficient, with a computational complexity of  $O(k \cdot n)$ . Instead, based on the above derivation, if we do not directly use the  $G_d$  matrix but instead employ a recursive approach to encode  $\mathbf{m}_d$ , the basic idea is to split m into two parts  $m_l$  and  $m_r$ , then encode  $m_l$  and  $m_r$  separately using  $G_{d-1}$ , and finally concatenate the encoded  $C_l$  and  $C_r$ .

$$\mathbf{w}_{d} = \left(m_{l}G_{d-1} + \operatorname{diag}(T_{d}) \circ m_{r}G_{d-1}\right) \parallel \left(m_{l}G_{d-1} + \operatorname{diag}(T_{d}') \circ m_{r}G_{d-1}\right)$$
(18)

In this way, the encoding process using  $G_d$  is transformed into two encoding processes using  $G_{d-1}$ . We can continue to recursively compute  $m_l G_{d-1}$  and  $m_r G_{d-1}$  until the split message length meets  $k_0$ , after which we simply use the generator matrix  $G_0$  for the base encoding. The encoding time complexity at this stage is  $O(k_0 \cdot n_0)$ . With a total of d recursive rounds, the overall computational effort is reduced to  $O(n \log(n))$ .

## RS code (foldable)

The RS code used in the FRI protocol satisfies the aforementioned conditions, making RS code foldable.

First, let us examine the generator matrix in the RS code:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{k-1} & \omega^{2(k-1)} & \cdots & \omega^{(n-1)(k-1)} \end{bmatrix}$$
(19)

where  $\omega$  is an *n*-th root of unity, satisfying  $\omega^n = 1$ , with *n* being the codeword length and *k* the message length. We can observe that the above matrix is a Vandermonde matrix. We will now explain how the generator matrix of the RS code satisfies the definition of Foldable Codes. For demonstration purposes, assume  $k_0 = 1$ ,  $n_0 = 1$ , d = 3,  $k_3 = 8$ ,  $n_3 = 8$ , and  $\omega^8 = 1$ ,

$$G^{(RS)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$
(20)

First, we need to reorder the rows of  $G^{RS}$  according to the Reversed Bit Order (RBO) sequence. The reason for performing RBO reordering is related to the structure of Multiplicative FFTs, as explained in another article. The RBO sequence refers to reversing the binary representation of the indices and using these reversed binary numbers as the new indices. The RBO sequence is (0, 4, 2, 6, 1, 5, 3, 7). By reordering the rows of the above Vandermonde matrix according to the RBO sequence, we obtain the following matrix, denoted as  $G_2$ :

$$G_{2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega & \omega^{6} & \omega^{3} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$(21)$$

Observing the reordered matrix, we can see that this matrix can be decomposed into submatrix operations involving  $G_1$  and  $T_1$ ,  $T'_1$ :

$$G_{2} = \begin{bmatrix} G_{1} & G_{1} \\ G_{1}T_{1} & G_{1}T_{1}' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega & \omega^{6} & \omega^{3} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$
(22)

where  $G_1$  is:

$$G_{1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} \end{bmatrix}$$
(23)

We can verify that the bottom-left submatrix can be expressed as  $G_1T_1$ :

$$G_{1}T_{1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \omega^{2} & & \\ & & & \omega^{3} \end{bmatrix} = \begin{bmatrix} 1 & \omega & \omega^{2} & \omega^{3} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} \end{bmatrix}$$
(24)

Here, the matrix  $T_1$  is indeed a diagonal matrix satisfying the requirement that its diagonal elements are distinct. The bottom-right submatrix can be decomposed as follows:

$$\begin{bmatrix} \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ \omega^{4} & \omega & \omega^{6} & \omega^{3} \\ \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ \omega^{4} & \omega^{3} & \omega^{2} & \omega^{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} \end{bmatrix} \begin{bmatrix} \omega^{4} & & & \\ & \omega^{5} & & \\ & & \omega^{6} & \\ & & & \omega^{7} \end{bmatrix} = G_{1}T_{1}'$$
(25)

where  $T_1'$  is exactly  $(-1)T_1$ , ensuring that the elements of  $T_1'$  are distinct from those of  $T_1$ :

$$T_{2}' = \begin{bmatrix} \omega^{4} & & & \\ & \omega^{5} & & \\ & & \omega^{6} & \\ & & & \omega^{7} \end{bmatrix} = \begin{bmatrix} -1 & & & & \\ & -\omega & & \\ & & -\omega^{2} & & \\ & & & -\omega^{3} \end{bmatrix} = -T_{2}$$
(26)

Next, we can continue to recursively decompose  $G_1$ :

$$G_{1} = \begin{bmatrix} 1 & 1 & | & 1 & 1 \\ 1 & \omega^{4} & | & 1 & \omega^{4} \\ 1 & \omega^{2} & | & \omega^{4} & \omega^{6} \\ 1 & \omega^{6} & | & \omega^{4} & \omega^{2} \end{bmatrix} = \begin{bmatrix} G_{0} & G_{0} \\ G_{0} \cdot T_{0} & G_{1} \cdot T_{0}' \end{bmatrix}$$
(27)

Here,  $G_0$ ,  $T_0$ , and  $T_0'$  are:

$$G_0 = \begin{bmatrix} 1 & 1 \\ 1 & \omega^4 \end{bmatrix}, \quad T_0 = \begin{bmatrix} 1 & \\ & \omega^2 \end{bmatrix}, \quad T'_0 = \begin{bmatrix} \omega^4 & \\ & \omega^6 \end{bmatrix} = -T_0$$
(28)

By recursively decomposing, we reach the base encoding  $G_0$ , which remains an RS-Code, denoted as  $RS[k_0 = 1, n_0 = 1]$ . Thus, we have proven that RS-Code is a foldable code. However, it is evident that RS-Code imposes requirements on  $\mathbb{F}_p$ , needing it to contain a multiplicative subgroup of order  $2^s$ , where s is a positive integer, and ensuring that s is large enough to accommodate a codeword length of  $k_0 \cdot 2^{n_d}$ .

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